ISRAEL JOURNAL OF MATHEMATICS **120** (2000), 449–466

HOLOMORPHY OF RANKIN TRIPLE L-FUNCTIONS; SPECIAL VALUES AND ROOT NUMBERS FOR SYMMETRIC CUBE L-FUNCTIONS

BY

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ABSTRACT

In this paper we prove the holomorphy of Rankin triple L-functions for three cusp forms on GL(2) on the entire complex plane, if at least one of them is non-monomial. We conclude the paper by proving the equality of our root numbers for the third and the fourth symmetric power L-functions with those of Artin through the local Langlands correspondence. We also revisit Deligne's conjecture on special values of symmetric cube L-functions.

* Partially supported by NSF grant DMS9610387.

** Partially supported by NSF grant DMS9970156.

Received December 14, 1998 and in revised form February 15, 2000

1. Introduction

The purpose of this paper is to give a short proof of the holomorphy of the completed Rankin triple L-function for three cuspidal representations of GL_2. The proof follows the same approach as the one in [Ki-Sh], where we proved the same result for the symmetric cube L-function for a non-monomial cusp form on GL_2. We conclude the paper by proving Deligne's conjecture [D] for the special values of the symmetric cube L-function at its critical values, as well as prove that the root numbers attached to the third and the fourth symmetric powers of a cusp form on GL_2 are Artin factors via local Langlands conjecture [Ku, La3].

More precisely, let π_i, $i = 1, 2, 3$, be three cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$, where $\mathbb{A} = \mathbb{A}_F$ is the ring of adeles of a number field F. For each i, write $\pi_i = \otimes \pi_{iv}$. Let S be a finite set of places for which all π_{iv}'s, $i = 1, 2, 3$, are unramified, whenever $v \notin S$. Let

$$(\mathrm{diag}(\alpha_{1v}, \alpha_{2v}),\ \mathrm{diag}(\beta_{1v}, \beta_{2v}),\ \mathrm{diag}(\gamma_{1v}, \gamma_{2v})) \in \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$$

denote representatives for the three semisimple conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$ which are attached to π_{iv}, $i = 1, 2, 3$, for each $v \notin S$. The local component of the Rankin triple L-function at each $v \notin S$ is simply given by

$$L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}) = \prod_{i,j,k=1,2} (1 - \alpha_{iv}\beta_{jv}\gamma_{kv}q_v^{-s})^{-1}.$$

Let

$$L_S(s,\ \pi_1 \times \pi_2 \times \pi_3) = \prod_{v \notin S} L(s,\ \pi_{1v} \times \pi_{2v} \times \pi_{3v}).$$

Using the split group Spin(8) and its Levi subgroup isomorphic to

$$(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{GL}_1)/\{\pm 1\},$$

the second author defined a local L-function and a local root number at each $v \in S$, canonically as to agree with local Langlands correspondence, and proved the standard functional equations that the completed triple product L-function satisfies, all in the context of the Langlands–Shahidi method [Sh1–7].

It was Garrett [Ga] who first found an integral representation for $L_S(s, \pi_1 \times \pi_2 \times \pi_3)$ for three holomorphic forms. This was extended to arbitrary cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ by Piatetski-Shapiro and Rallis [PS-Ra]. On the other hand, using [PS-Ra], Ikeda [Ik] defined local L-functions $L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v})$ at every $v \in S$, and using an idea of Blasius determined the poles of the completed L-function. In particular, he showed that it does not have a pole unless all of the π_i's are monomial, which simply means that π_i is stable under twist by a nontrivial character of $F^*\backslash\mathbb{A}^*$.

In a recent paper, Ramakrishnan [Ram2] completed the whole process of integral representations and proved that the local factors defined either by this method, that of Langlands–Shahidi discussed above, or the one coming from representations of the Deligne–Weil group parametrizing π_{iv}'s are all the same. (He in fact proved much more and, in particular, the modularity of $\pi_1 \times \pi_2$.)

In this paper we use our own method [Sh1,Ki3,Ki-Sh] to prove the holomorphy of $L(s, \pi_1 \times \pi_2 \times \pi_3)$ on all of $\mathbb{C}$, where at least one of the representations is not monomial. Our method is quite effective and avoids a lot of local analysis which is necessary in the method of integral representations. The local analysis is only that of local intertwining operators for which systematic and reasonably general results are now available (cf. [C-Sh], [Ki3], [Ki-Sh], [Za], [Zh]).

Because of the functional equation, we only have to prove the holomorphy for $\mathrm{Re}(s) \geq \frac{1}{2}$. Without loss of generality, we can and will assume that poles of the L-functions are on the real axis (see section 3). We have to divide into four cases: $\frac{1}{2} < s < 1, s > 1$, $s = \frac{1}{2}$, and $s = 1$. As in [Ki-Sh], $s = \frac{1}{2}$ is done using quadratic base change. For the case $\frac{1}{2} < s < 1, s > 1$, we closely follow [Ki-Sh], using Ramakrishnan's result [Ram1] and some results on unitary representations [Li]. For $s = 1$, we use the fact that the local L-functions are holomorphic for $\mathrm{Re}\, s \geq \frac{3}{4}$ ($\mathrm{Re}\, s \geq 1$ is enough) and thus the completed L-functions and Ikeda's definition of L-functions have poles at the same place for $\mathrm{Re}\, s \geq 1$ (or use [Ram2]). We then use Ikeda's result.

We should note that the same method would also apply to twisted triple L-functions (Cases ${}^2D_4 - 1$, ${}^3D_4 - 1$, and ${}^6D_4 - 1$ in [Sh3]). We leave this to a future paper.

We conclude the paper by proving Deligne's conjecture [D, Z] on the special critical values of symmetric cube L-functions for holomorphic forms on GL(2). We do this using the results of Garrett and Harris [GH] in which they prove Blasius's interpretation [B] of Deligne's conjecture for the triple product L-function, combined with Shimura [S]. We refer to Orloff [O] for level 1 results of [GH]. Beside a careful checking of special value results established in [B, GH, S], which in one form or other should be known to experts (cf. Theorem 6.3 of [GH]), it is the finiteness of these values [Ki-Sh] which makes our Theorem 4.1 new and deep. Finally, in Theorem 4.2 we show that the local root numbers for the third and fourth symmetric power L-functions defined in [Sh5, Sh7] are in fact those of Artin attached to their parametrizations by means of the local Langlands conjecture [Ku, La3], using the recent important proof of the local Langlands conjecture for GL_n by Harris and Taylor [HT].

The second author wishes to thank E. de Shalit, J. Rogawski, and P. Schneider for their kind invitation to their conference: "p-adic aspects of the theory of automorphic representations," February 98, Jerusalem, in connection with which this manuscript was prepared.

2. Some facts on unitary representations

In this section, we assume that F is a local field of characteristic 0. We restrict ourselves to the case of split reductive groups. Let χ be an unramified unitary character of T and $\Lambda \in \mathfrak{a}^* = X(T)_F \otimes \mathbb{R}$ and $\chi' = \Lambda \otimes \chi$. Then the induced representation $I(\Lambda, \chi) = \mathrm{Ind}_B^G\ \chi'$ is defined. It has a unique unramified irreducible subquotient, denoted by $\pi(\Lambda, \chi)$. Suppose Λ is in the closed positive Weyl chamber and let $\Delta_1 = \{\alpha \in \Delta |\ \Lambda \circ \alpha^\vee = 1\}$. Let $P_1 = M_1 N_1$ be the standard parabolic subgroup of G generated by the roots in Δ_1. Let π_1 be the unique irreducible spherical subrepresentation of $\mathrm{Ind}_{B\cap M_1}^{M_1}\ \chi$.

THEOREM 2.1 ([Li, Theorem 2.2, page 749]): *The following are equivalent:*

(1) $\chi' \circ \alpha^\vee \neq |\ |$ *for any* α,

(2) $\mathrm{Ind}_{P_1}^G\ \Lambda \otimes \pi_1$ *is irreducible (hence equals* $\pi(\Lambda, \chi)$*),*

(3) $\pi(\Lambda, \chi)$ *is generic.*

PROPOSITION 2.2 ([Li, Lemma 2.3, page 751]): *Let $\tilde{G}$ and G be unramified reductive groups over F, and let $\phi : \tilde{G} \longmapsto G$ be a central isogeny defined over F. Let $\tilde{B} = \tilde{T}\tilde{U}$ be a Borel F-subgroup of $\tilde{G}$ and assume ϕ maps $\tilde{B}, \tilde{T}, \tilde{G}(\mathcal{O})$ to B, T and $G(\mathcal{O})$, respectively. Let χ be an unramified unitary character of T. Then we can define a unitary character $\tilde{\chi} = \phi^*(\chi)$ of $\tilde{T}$ by $\tilde{\chi}(\tilde{t}) = \chi(\phi(\tilde{t}))$. Conversely, given any $\tilde{\chi}$, there will be finitely many χ such that $\tilde{\chi} = \phi^*(\chi)$. Then $\pi(\Lambda, \tilde{\chi})$ is unitary if and only if $\pi(\Lambda, \chi)$ is.*

3. Main results

Let π_i, $i = 1, 2, 3$, be cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ with central characters ω_i, $i = 1, 2, 3$. Let $\omega = \omega_1\omega_2\omega_3$. Let π_{iv}, $i = 1, 2, 3$, be unramified for $v \notin S$, where S is a finite set of places including archimedean places. Then $\pi_{iv} = \pi(\mu_{iv}, \nu_{iv})$, $i = 1, 2, 3$, where μ_{iv}, ν_{iv} are unramified quasi-characters of $F_v^\times$. Let $\alpha_{1v} = \mu_{1v}(\varpi)$, $\alpha_{2v} = \nu_{1v}(\varpi)$, $\beta_{1v} = \mu_{2v}(\varpi)$, $\beta_{2v} = \nu_{2v}(\varpi)$, $\gamma_{1v} = \mu_{3v}(\varpi)$,

$\gamma_{2v} = \nu_{3v}(\varpi)$. Then the local Rankin triple L-function is defined to be

$$L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}, \rho_2 \otimes \rho_2 \otimes \rho_2) = \prod_{i,j,k=1,2} (1 - \alpha_{iv}\beta_{jv}\gamma_{kv}q_v^{-s})^{-1},$$

where ρ_2 is the standard representation of $\mathrm{GL}_2(\mathbb{C})$. Let

$$L_S(s, \pi_1 \times \pi_2 \times \pi_3, \rho_2 \otimes \rho_2 \otimes \rho_2) = \prod_{v \notin S} L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}, \rho_2 \otimes \rho_2 \otimes \rho_2).$$

For simplicity, we set $L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}) = L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}, \rho_2 \otimes \rho_2 \otimes \rho_2)$ and $L_S(s, \pi_1 \times \pi_2 \times \pi_3) = L_S(s, \pi_1 \times \pi_2 \times \pi_3, \rho_2 \otimes \rho_2 \otimes \rho_2)$.

Let $G = \mathrm{Spin}(8)$ and $\theta = \{\alpha_1 = e_1 - e_2, \alpha_3 = e_3 - e_4, \alpha_4 = e_3 + e_4\}$. Let $T \subset M_\theta = M$ be the Levi subgroup of G generated by θ and let $P = MN$ be the corresponding standard parabolic subgroup of G. Then standard calculations show that $M = (\mathrm{GL}_1 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2)/\{\pm 1\}$, where $-1 = (-1, -I_2, -I_2, -I_2)$. More precisely, $M = A \times M_D/(A \cup M_D)$, where $M_D = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and $A \simeq \mathrm{GL}_1$ is the connected component of the center of M;

$$A = \{H_{\alpha_1}(t)H_{\alpha_2}(t^2)H_{\alpha_3}(t)H_{\alpha_4}(t) \colon t \in \overline{F}^*\},$$
$$A \cap M_D = \{H_{\alpha_1}(t)H_{\alpha_2}(t^2)H_{\alpha_3}(t)H_{\alpha_4}(t) \colon t^2 = 1\}.$$

Let π_{i0}, $i = 1, 2, 3$, denote constituents of $\pi_i|_{\mathrm{SL}_2(\mathbb{A}_F)}$, resp. Then $\sigma = \omega_1\omega_2\omega_3 \otimes \pi_{10} \otimes \pi_{20} \otimes \pi_{30}$ is a cuspidal representation on $M(\mathbb{A}_F)$. Denote by ${}^LM$ the L-group of $\mathbf{M}$ and let ${}^L\mathfrak{n}$ be the Lie algebra of the L-group of $\mathbf{N}$. Let r be the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$. Then we have [Sh4]

$$r = r_1 \oplus r_2$$
$$L(s, \sigma_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v})$$
$$L(s, \sigma_v, r_2) = L(s, \omega_v)$$

For each $v \in S$, Shahidi [Sh1, Sh6] defined the local L-function $L(s, \sigma_v, r_1)$. It is defined to agree completely with Langlands' definition of L-functions defined in terms of parametrization. In particular, the L-function for arbitrary σ_v is just the analytic continuation of the one attached to the tempered inducing data through the product formula (cf. part 3 of Theorem 3.5 and equation 7.10 of [Sh1]). Denote by $L(s, \pi_1 \times \pi_2 \times \pi_3)$ the completed triple L-function

$$L(s, \pi_1 \times \pi_2 \times \pi_3) = \prod_{\text{all } v} L(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}).$$

THEOREM 3.1 ([Sh1]): *The L-function $L(s, \pi_1 \times \pi_2 \times \pi_3)$ can be continued to a meromorphic function on the whole complex plane and satisfies the standard functional equation $L(s, \pi_1 \times \pi_2 \times \pi_3) = \epsilon(s, \pi_1 \times \pi_2 \times \pi_3) L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2 \times \tilde{\pi}_3)$.*

Because of this theorem, it is enough to consider the holomorphy for $\mathrm{Re}\, s \geq \frac{1}{2}$. We assume that the central character of σ, i.e., ω, is trivial on F_∞^+, where $\mathbb{A}_F^* = \mathbb{I}^1 \cdot F_\infty^+$ with $\mathbb{I}^1$ ideles of norm 1, so that the poles of Eisenstein series may be on the real axis. Then, the poles of the Eisenstein series attached to (M, σ) coincide with those of its constant term. Let α be the unique simple root in $\mathbf{N}$. As in [Sh1], let $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \cdot \rho$, where ρ is half the sum of roots in $\mathbf{N}$. We identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}_{\mathbb{C}}^*$ and denote $I(s, \sigma) = I(s\tilde{\alpha}, \sigma) = \mathrm{Ind}_P^G\, \sigma \otimes \exp(\langle s\tilde{\alpha}, H_P(\ )\rangle)$. Let $A(s\tilde{\alpha}, \sigma, w_0)$ be the standard intertwining operator from $I(s\tilde{\alpha}, \sigma)$ into $I(w_0(s\tilde{\alpha}), w_0(\sigma))$. For $f \in I(s, \sigma)$, let $E(s, f, g, P)$ be the Eisenstein series attached to (M, σ). Then the constant term of $E(s, f, g, P)$ along N is given by

$$E_N(s, f, g, P) = f(g) + M(s, \sigma, w_0) f(g),$$

where $M(s, \sigma, w_0) = \otimes_v A(s, \sigma_v, w_0)$ ([La1, La2, M-W1, Ki1, Ki2]). We normalize the intertwining operator $A(s, \sigma_v, w_0)$ as follows [Sh1]:

$$A(s, \sigma_v, w_0) = r(s, \sigma_v, w_0) N(s, \sigma_v, w_0),$$
$$r(s, \sigma_v, w_0) = \prod_{i=1}^{2} \frac{L(is, \sigma_v, \tilde{r}_i)}{L(1+is, \sigma_v, \tilde{r}_i)\epsilon(s, \sigma_v, \tilde{r}_i, \psi_v)}.$$

Let $N(s, \sigma, w_0) = \otimes_v N(s, \sigma_v, w_0)$, $r(s, \sigma, w_0) = \prod_v r(s, \sigma_v, w_0)$ and $\epsilon(s, \sigma, r_i) = \prod_v \epsilon(s, \sigma_v, r_i, \psi_v)$. Then we have, for $f \in I(s, \sigma)$,

$$M(s, \sigma, w_0) f = r(s, \sigma, w_0) N(s, \sigma, w_0) f, \quad r(s, \sigma, w_0) = \prod_{i=1}^{2} \frac{L(is, \sigma, \tilde{r}_i)}{L(1+is, \sigma, \tilde{r}_i)\epsilon(s, \sigma, \tilde{r}_i)}.$$

PROPOSITION 3.2: *The normalized intertwining operator $N(s, \sigma_v, w_0)$ is holomorphic and non-zero for $\mathrm{Re}\, s > 0$.*

Proof: If σ_v is tempered, then the unnormalized intertwining operator $A(s, \sigma_v, w_0)$ is holomorphic and non–zero for $\mathrm{Re}(s) > 0$, while the local L-function $L(s, \sigma_v, r_1)$ is holomorphic there since Conjecture 7.1 of [Sh1, Proposition 7.2] applies.

If σ_v is non-tempered, we write $I(s, \sigma_v)$ as in [Ki3, p. 481],

$$I(s, \sigma_v) = I(s\tilde{\alpha} + \Lambda_0,\ \pi_0) = \mathrm{Ind}_{M_0(F_v)N_0(F_v)}^{G(F_v)} \pi_0 \otimes q^{\langle s\tilde{\alpha} + \Lambda_0, H_{P_0}(\ )\rangle} \otimes \mathbf{1},$$

where π_0 is a tempered representation of $M_0(F_v)$ and $P_0 = M_0N_0$ is another parabolic subgroup of G. We can identify the normalized operator $N(s, \sigma_v, w_0)$ with the normalized operator $N(s\tilde{\alpha}+\Lambda_0,\ \pi_0, \tilde{w}_0)$, which is a product of rank-one operators attached to tempered representations (cf. [Zh, Proposition 1]).

If σ_v is non-tempered, we can easily check that these rank one operators are operators for either a minimal parabolic inside a group whose derived group is SL_2; a parabolic subgroup whose Levi subgroup has a derived group isomorphic to SL_2 inside a group whose derived group is SL_3; or isomorphic to $SL_2 \times SL_2$ inside a group with SL_4 as its derived group.

These operators are then restrictions, to SL_2, SL_3, and SL_4, respectively, of corresponding standard operators for GL_n, $n = 2, 3, 4$.

From [M-W2, Proposition I.10] one knows that these rank one operators are holomorphic for $\mathrm{Re}(s) > -1$. Thus we only need to check that $\mathrm{Re}(\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee\rangle) > -1$ for all positive roots of G with respect to A_{θ_0} and for $\mathrm{Re}(s) > 0$, where $M_0 = M_{\theta_0}$, $\theta_0 \subset \theta$.

Since G is simply connected, we can embed our three copies of SL_2 in G. We can therefore write

$$s\tilde{\alpha} + \Lambda_0 = (s + r_1)e_1 + (s - r_1)e_2 + (r_2 + r_3)e_3 + (r_2 - r_3)e_4,$$

where $r_i = 0$ if π_{iv} is tempered, and $0 < r_i < 1/4$ if π_{iv} is non-tempered. Consequently if $\mathrm{Re}(s) > 0$, then $\mathrm{Re}(s) - r_1 - (r_2 + r_3) > 0$.

Ranging β over positive roots of G with respect to T, realized as roots of A_{θ_0}, one determines that the least value of $\mathrm{Re}(\langle s\tilde{\alpha}+\Lambda_0, \beta^\vee\rangle)$ is $\mathrm{Re}(s) - r_1 - (r_2 + r_3)$ which, as we observed above, is larger than -1. Consequently, $N(s\tilde{\alpha}+\Lambda_0, \pi_0, \tilde{w}_0)$ is holomorphic for $\mathrm{Re}(s) \geq 0$. By Zhang's Lemma (cf. [Ki4, Lemma 1.7]) it is non-zero as well. ∎

Remark: In this paper we only need the statement of Proposition 3.2 for $\mathrm{Re}(s) \geq 1/2$.

Denote the image of $N(s, \sigma_v, w_0)$ by $J(s, \sigma_v)$. If σ_v is tempered, it is the usual Langlands' quotient $J(s, \sigma_v)$. But if σ_v is non-tempered, it is a Langlands' quotient from smaller parabolic subgroups. Let $J(s, \sigma) = \otimes_v J(s, \sigma_v)$. Recall the following [Ki4, Observation 1.3]:

LEMMA 3.3: *If $r(s, \sigma, w_0)$ has a pole for $s > 0$, then $J(s, \sigma) = \otimes_v J(s, \sigma_v)$ belongs to the residual spectrum $L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(M,\sigma)}$ and, in particular, each $J(s, \sigma_v)$ is unitary.*

LEMMA 3.4: *Let F be a local field. Let $\mu_i, i=1,2,3,4$, be unitary unramified characters of $F^\times$ and let $\pi_1 = \mathrm{Ind}_B^{\mathrm{GL}_2}\ \mu_1\times\mu_2$. We assume that π_1 has trivial central character, i.e., $\mu_1\mu_2=1$. Let π_2 be the unique generic component of $\mathrm{Ind}_B^{\mathrm{SO}_4}\ \mu_3\times\mu_4$. Then if $s>\frac12$, $s\neq 1$, $I=\mathrm{Ind}_{\mathrm{GL}_2\times\mathrm{SO}_4}^{\mathrm{SO}_8}\ |\det|^s\pi_1\times\pi_2$ is irreducible but not unitary.*

Proof: By Theorem 2.1, if $s>\frac12$, $s\neq 1$, I is irreducible. Therefore I cannot be unitary for $s>1$. Suppose $\frac12<s<1$ and I is unitary. Since $\mu_2=\mu_1^{-1}$,

$$\begin{aligned}I \simeq \mathrm{Ind}_{F^\times\times F^\times\times\mathrm{SO}_4}^{\mathrm{SO}_8}\ |\ |^s\mu_1\times|\ |^s\mu_1^{-1}\times\pi_2 &\simeq \mathrm{Ind}_{F^\times\times F^\times\times\mathrm{SO}_4}^{\mathrm{SO}_8}\ |\ |^s\mu_1\times|\ |^{-s}\mu_1\times c_4(\pi_2)\\ &\simeq \mathrm{Ind}_{\mathrm{GL}_2\times\mathrm{SO}_4}^{\mathrm{SO}_8}\ \pi(|\ |^s\mu_1\times|\ |^{-s}\mu_1)\times c_4(\pi_2),\end{aligned}$$

where c_4 is the sign change. Note that $\pi(|\ |^s\mu_1\times|\ |^{-s}\mu_1)$ is hermitian for all s and therefore, by [Mu, Lemma 5.1], $\pi(|\ |^s\mu_1\times|\ |^{-s}\mu_1)$ must be unitary. This is a contradiction since this last representation is not unitary for $s>\frac12$. ∎

LEMMA 3.5: *Let π_1,π_2,π_3 be three cuspidal representations of GL_2. Let T be the set of places where $\pi_{1v},\pi_{2v},\pi_{3v}$ are all tempered. Then $\underline{\delta}(T)\geq\frac{7}{10}$.*

Proof: Let T_i be the set of places where π_{iv} is tempered for $i=1,2,3$. Then from [Ram1], $\underline{\delta}(T_i)\geq\frac{9}{10}$ for $i=1,2,3$. Let X be the set of all places. Then $\bar\delta(T_1-T_2)\leq\bar\delta(X-T_2)\leq\frac{1}{10}$. We have $T_1=(T_1\cap T_2)\cup(T_1-T_2)$. So $\underline{\delta}(T_1\cap T_2)\geq\underline{\delta}(T_1)-\bar\delta(T_1-T_2)\geq\frac{9}{10}-\frac{1}{10}=\frac{8}{10}$. Similarly, we have $\underline{\delta}(T_1\cap T_2\cap T_3)\geq\frac{7}{10}$. ∎

LEMMA 3.6: *The local L-function $L(s,\pi_{1v}\times\pi_{2v}\times\pi_{3v})$ is holomorphic for $\mathrm{Re}\, s\geq\frac34$.*

Proof: The first statement follows from the definition of the local L-functions and Gelbart–Jacquet's estimates on Fourier coefficients [Ge-Ja]; $\pi_v=\pi(\mu|\ |^r,\mu|\ |^{-r}), r<\frac14$ for any v, unramified or ramified. ∎

LEMMA 3.7: *Suppose $r(s,\sigma,w_0)$ has a pole for $s\geq\frac12$. Then $\omega^2=1$.*

Proof: Suppose σ_v is tempered. Then by assumption, $J(s,\pi_v)$ is unitary and therefore $J(s,\pi_v)$ is hermitian. So we have $\omega_v^2=1$. By Lemma 3.5, $\omega_v^2=1$ for a set of primes whose lower Dirichlet density is at least 7/10. One can now apply the result of Hecke that two idele class characters agreeing at all the places in a set of Dirichlet density larger than 1/2 are equal, to conclude that $\omega^2=1$. ∎

PROPOSITION 3.8: *Under the assumption that ω is trivial on F_∞^+, the possible poles of the triple L-function $L(s, \pi_1 \times \pi_2 \times \pi_3)$ for* $\mathrm{Re}(s) \geq 1/2$ *are at* $s = 1/2, 1$.

Proof: By Lemma 3.7, we can assume $\omega^2 = 1$. Then $\omega_v = 1$ for a set of places of density $\frac{1}{2}$. By Lemma 3.5, there exists a place v where π_{iv}, $i = 1, 2, 3$, are tempered, unramified, and $\omega_v = 1$. Let $\pi_{iv} = \pi(\mu_i, \nu_i)$, $i = 1, 2, 3$. Then $\omega_{iv} = \mu_i\nu_i$. Under the isogeny $\mathrm{Spin}(8) \longmapsto \mathrm{SO}_8$, $I(s, \sigma_v)$ corresponds to an induced representation

$$\mathrm{Ind}_{\mathrm{GL}_2 \times \mathrm{SO}_4}^{\mathrm{SO}_8} |\det|^s \pi_1 \times \pi_2,$$

where $\pi_1 = \mathrm{Ind}_B^{\mathrm{GL}_2} \eta_1 \times \eta_2$, $\eta_1^2 = \mu_1^2\omega_2\omega_3$, $\eta_2^2 = \nu_1^2\omega_2\omega_3$, and π_2 is the unique generic component of $\mathrm{Ind}_B^{\mathrm{SO}_4} \eta_3 \times \eta_4$, $\eta_3^2 = \mu_2\nu_2^{-1}\mu_3\nu_3^{-1}$, and $\eta_4^2 = \mu_2^{-1}\nu_2\mu_3\nu_3^{-1}$.

By Proposition 2.2, $J(s, \sigma_v)$ is not unitary if $s > 1/2$, $s \neq 1$. Hence by Proposition 3.2 and Lemma 3.3, $r(s, \sigma, w_0)$ does not have a pole for s in the same range. Using this fact, the definition of $r(s, \tilde{\sigma}, w_0)$, and starting with $\mathrm{Re}(s)$ large where both L-functions converge absolutely, one can argue inductively that $L(s, \pi_1 \times \pi_2 \times \pi_3)L(2s, \omega)$ is holomorphic for $\mathrm{Re}(s) > 1$, appealing to non-vanishing of $L(s, \omega)$ for $\mathrm{Re}(s) > 1$ in each step. Since $L(2s, \omega)$ is in fact non-zero for $\mathrm{Re}(s) \geq 1/2$, we conclude that the poles of $r(s, \tilde{\sigma}, w_0)$ for $\mathrm{Re}(s) \geq 1/2$ are those of $L(s, \pi_1 \times \pi_2 \times \pi_3)$. This implies that poles of $L(s, \pi_1 \times \pi_2 \times \pi_3)$ are all real and possibly only at $s = 1/2$ and 1 if $\mathrm{Re}(s) \geq 1/2$. ∎

PROPOSITION 3.9: *The triple L-function does not have a pole at* $s = \frac{1}{2}$.

Proof: By Lemma 3.7, $\omega^2 = 1$. If $\omega = 1$, then the triple L-function does not have a pole at $s = \frac{1}{2}$ because the Eisenstein series has only simple poles and the pole is already that of $L(2s, \omega)$ at $s = \frac{1}{2}$. Suppose $\omega^2 = 1, \omega \neq 1$. Let E/F be the quadratic extension defined by ω via class field theory. Denote by Π_i the base change of π_i to $\mathrm{GL}_2(\mathbb{A}_E)$. Then

$$L(s, \Pi_1 \times \Pi_2 \times \Pi_3) = L(s, \pi_1 \times \pi_2 \times \pi_3)L(s, (\pi_1 \otimes \omega) \times \pi_2 \times \pi_3). \tag{3.9.1}$$

It is enough to prove (3.9.1) locally. It clearly holds when $v = \infty$ or π_{iv}'s are all unramified, $i = 1, 2, 3$. Next we assume all π_{iv}'s are supercuspidal and apply Proposition 5.1 and functional equation (3.14) of [Sh1] to conclude, up to multiplication by appropriate λ-functions, the equality

$$\begin{aligned} \gamma(s, \Pi_{1w} \times \Pi_{2w} \times \Pi_{3w}, \psi_{E_w/F_v}) =& \gamma(s, \pi_{1v} \times \pi_{2v} \times \pi_{3v}, \psi_{F_v}) \\ &\cdot \gamma(s, (\pi_{1v} \otimes \omega_v) \times \pi_{2v} \times \pi_{3v}, \psi_{F_v}), \end{aligned} \tag{3.9.2}$$

where ψ_{F_v} is an additive character and $\psi_{E_w/F_v} = \psi_{F_v} \cdot \mathrm{Tr}_{E_w/F_v}$, $w|v$.

Next assume π_{iv}'s are tempered, but not all supercuspidal. Then (3.9.2) is a consequence of multiplicativity of γ-functions (part 3 of Theorem 3.5 of [Sh1]) and similar identities for Rankin–Selberg γ-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$ and $\mathrm{GL}_1 \times \mathrm{GL}_2$. The local version of equality (3.9.1) in the tempered case then follows immediately from (3.9.2) since L-functions in [Sh1] are defined by means of zeros of γ-functions (Section 7 of [Sh1]), when the representations are tempered. For arbitrary local representations, the L-functions are defined in [Sh1] by analytic continuation of tempered ones in terms of their Langlands parameters (page 308 of [Sh1]). This proves (3.9.1) in general.

It should be pointed out that parametrizations in [Ku], [HT], and [H2] are not enough to prove (3.9.1), since [HT] only guarantees equality of L-functions for Rankin–Selberg product L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$, but not $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$.

Now note that if π_i is not monomial [L-La], Π_i remains cuspidal. In this case, the left hand side of (3.9.1) is holomorphic at $s = \frac{1}{2}$ since $\omega_\Pi = 1$. Suppose one of π_i is monomial, say π_1. Then Π_1 is not cuspidal. Consequently, $\Pi_1 = \pi(\mu, \nu)$ for some grössencharacters μ and ν. Then

$$(3.9.3) \qquad L(s, \Pi_1 \times \Pi_2 \times \Pi_3) = L(s, (\Pi_2 \otimes \mu) \times \Pi_3) L(s, (\Pi_2 \otimes \nu) \times \Pi_3).$$

One only needs to prove (3.9.3) locally, to which again the same types of arguments as those for the proof of (3.9.1) apply. In fact, (3.9.3) is locally precisely the multiplicativity of L-functions which one can get from multiplicativity of γ-functions (part 3 of Theorem 3.5 of [Sh1]), since

$$\Pi_{1v} \otimes \Pi_{2v} \otimes \Pi_{3v} = \mathrm{Ind}[(\mu_v, \nu_v) \otimes \Pi_{2v} \otimes \Pi_{3v}].$$

At any rate $L(s, \Pi_1 \times \Pi_2 \times \Pi_3)$ is holomorphic at $s = 1/2$ in all cases.

Note that $\pi_i \otimes \omega_i = \tilde{\pi}_i$, and by the functional equation

$$L(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3) = \epsilon(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3) L(\frac{1}{2}, \tilde{\pi}_1 \times \tilde{\pi}_2 \times \tilde{\pi}_3).$$

Thus if $L(s, \pi_1 \times \pi_2 \times \pi_3)$ has a pole at $s = \frac{1}{2}$, then $L(s, \Pi_1 \times \Pi_2 \times \Pi_3)$ must have a double pole at $s = \frac{1}{2}$. This contradicts the holomorphy (in fact simplicity of the pole is enough) of $L(s, \Pi_1 \times \Pi_2 \times \Pi_3)$ at $s = \frac{1}{2}$, as the central character of Π is trivial. ∎

PROPOSITION 3.10 (Ikeda): *Suppose the triple L-function $L(s, \pi_1 \times \pi_2 \times \pi_3)$ has a pole at $s = 1$. Then $\omega^2 = 1$ and $\omega \neq 1$. Let E be the quadratic extension of F corresponding to ω by class field theory. Then there exist quasi-characters*

χ_1, χ_2, χ_3 *of* $\mathbb{A}_E^\times/E^\times$ *such that* $\pi_1 = \pi(\chi_1), \pi_2 = \pi(\chi_2), \pi_3 = \pi(\chi_3)$ *and* $\chi_1\chi_2\chi_3 = 1$.

Proof: By Lemma 3.7, $\omega^2 = 1$. By Lemma 3.6 and [Ik, Lemma 2.1], for $\mathrm{Re}\, s \geq 1$, the location of the poles of the completed triple L-functions and those of Ikeda's definition of the L-functions coincide. Therefore, if $\omega = 1$, the completed L-function is holomorphic at $s = 1$ by [Ik, Proposition 2.5]. If $\omega \neq 1$, we proceed exactly in the same way as in [Ik, Theorem 2.6], using the quadratic base change. ∎

Remark 3.1: We were unable to prove the holomorphy of the completed L-function at $s = 1$ when $\omega = 1$, without Ikeda's result [Ik, Proposition 2.5].

THEOREM 3.11: *The completed triple L-function* $L(s, \pi_1 \times \pi_2 \times \pi_3)$ *is holomorphic except possibly at* $s = 0, 1$.

Proof: This follows from the functional equation of the triple L-function and Propositions 3.8 and 3.9. ∎

4. Special values and root numbers for symmetric cube L-functions

Let φ be a normalized holomorphic new eigenform of (even) weight k and Nebentypus ε with respect to $\Gamma_1(N)$. Let E be the field generated over $\mathbb{Q}$ by Fourier coefficients of φ. We consider E as a subfield of complex numbers. We will write $a \sim b$ for two non-zero complex numbers a and b if $ab^{-1} \in E$.

Next, let c^+ and c^- be Deligne's periods of φ (cf. [D, Z]). In particular, they satisfy

$$L_f(\ell, \varphi) \sim (2\pi i)^\ell c^\pm,$$

where $\pm = (-1)^\ell$, $1 \leq \ell \leq k-1$. Here $L_f(s, \varphi)$ denotes the finite part of the L-function of φ, or the corresponding Dirichlet series. Moreover, let $G(\varepsilon)$ be the Gauss sum

$$G(\varepsilon) = \sum_{a=1}^{N} \varepsilon(a) e^{2\pi i a/N}$$

attached to ε: $(\mathbb{Z}/n\mathbb{Z})^* \longrightarrow \mathbb{C}^*$, and set (cf. [B, D])

$$\delta(\varphi) = (2\pi i)^{1-k} G(\varepsilon). \tag{4.1}$$

Recall that

$$L_f(s, \varphi) = \prod_p (1 - a_p p^{-s} + \varepsilon(p) p^{k-1-2s})^{-1},$$

where $\varepsilon(p)=0$ if $p|N$. Write

$$L_p(s,\varphi)^{-1}=1-a_pp^{-s}+\varepsilon(p)p^{k-1-2s}$$
$$=(1-\alpha'_pp^{-s})(1-\alpha''_pp^{-s}).$$

Then $\alpha'_p\alpha''_p=\varepsilon(p)p^{k-1}$ and we will understand that if $p|N$, then $\alpha''_p=0$ and $\alpha'_p=a_p$. We then define

$$L_p(s,\varphi\otimes\varepsilon)\overset{\text{def}^{\text{n}}}{=}(1-\alpha'^2_p\alpha''_pp^{-s})^{-1}(1-\alpha'_p\alpha''^2_pp^{-s})^{-1}.$$

Then

$$L_p(s,\varphi\otimes\varepsilon)=(1-\varepsilon(p)a_pp^{-(s-(k-1))}+\varepsilon(p)^2\varepsilon(p)p^{k-1-2(s-(k-1))})^{-1}$$
$$=L_p(s-(k-1),\varphi,\varepsilon),$$

where the last factor is the pth factor of $L_f(s-(k-1),\varphi,\varepsilon)$, the Dirichlet series attached to φ and ε. Now by Theorem 1 of Shimura [S]

$$L_f(m,\varphi\otimes\varepsilon)=L_f(m-(k-1),\varphi,\varepsilon)$$
$$\sim(2\pi i)^{m-(k-1)}c^{\pm(-1)^{m-(k-1)}}G(\varepsilon)$$

for $k\le m\le 2k-2$ or $1\le m-(k-1)\le k-1$. Here we use Shimura's sign description and the fact that $\varepsilon(-1)=1$. Thus

$$L_f(m,\varphi\otimes\varepsilon)\sim(2\pi i)^mc^{\mp(-1)^m}\delta(\varphi) \tag{4.2}$$

since k is even, using (4.1).

Next, let $L_f(s,\varphi\otimes\varphi\otimes\varphi)$ be the Dirichlet series of Garrett [Ga] attached to three copies of φ. Its set of critical values is then $k\le m\le 2k-2$. Applying Blasius's formula on the top of page 187 of [B] to $L_f(s,\varphi\otimes\varphi\otimes\varphi)$, thus letting $r=3$,

$$n_i=\frac{1}{2}\binom{r-1}{\frac{r-1}{2}}=1,$$

and $m_i=2^{r-2}-n_i=1$ (cf. §2.2 of [B]), one expects

$$L_f(m,\varphi\otimes\varphi\otimes\varphi)\sim(2\pi i)^{4m}(c^+c^-)^3\delta(\varphi)^3 \tag{4.3}$$

for every critical value m, i.e., integers $k\le m\le 2k-2$. When φ is of level one (i.e., with respect to $SL_2(\mathbb{Z})$), (4.3) was proved by Orloff [B, O]. On the other hand, the proof of (4.3) for arbitrary φ is the subject matter of [GH], §6.4 (cf. also [HK]). (For the whole interval, one needs to use the precise form of the functional equation which is now proved in full generality [Sh1]; cf. the introduction in [GH].)

We now recall the symmetric cube L-function attached to φ by Deligne [D] whose local factors are defined by

$$L_p(s, \mathrm{Sym}^3 \varphi) = \prod_{j=0}^{3} (1 - \alpha_p'^j \alpha_p''^{3-j} p^{-s})^{-1}.$$

In his notation $d^{\pm} = 2$ and he conjectures [D, Z] that

$$L_f(m, \mathrm{Sym}^3 \varphi) \sim (2\pi i)^{2m} (c^{\pm})^3 c^{\mp} \delta(\varphi),$$

where $\pm = (-1)^m$ for all critical values m, $k \leq m \leq 2k-2$. Implicit in it is that $L_f(m, \mathrm{Sym}^3 \varphi)$ is finite for all such m. This was proved in [Ki-Sh] and it is this which is new in the following proposition, as the rest of the material in this section is surely well known to experts in one form or another. (See Theorem 6.2 of [GH].)

PROPOSITION 4.1: *Let $k \leq m \leq 2k-2$ denote the set of critical values m for $L_f(s, \mathrm{Sym}^3 \varphi)$. Then each $L_f(m, \mathrm{Sym}^3 \varphi)$ is finite and Deligne's conjecture*

$$L_f(m, \mathrm{Sym}^3 \varphi) \sim (2\pi i)^m (c^{\pm})^3 c^{\mp} \delta(\varphi)$$

is valid, where $\pm = (-1)^m$.

Proof: We only need to observe that

$$L_f(m, \mathrm{Sym}^3 \varphi) = L_f(m, \varphi \otimes \varphi \otimes \varphi) / L_f(m, \varphi \otimes \varepsilon)^2$$

and use (4.2) and (4.3).

Next we turn our attention to the root numbers attached to symmetric cube L-functions for GL_2. We shall show that they are the same as their Artin counterparts [La3, Sh6, T].

Let F be a local field of characteristic zero, archimedean or otherwise. Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$. Fix a non-trivial additive character ψ of F. In [Sh5], we attached a root number $\varepsilon(s, \pi, \mathrm{Sym}^3(\rho_2), \psi)$ which further satisfied

$$\varepsilon(s, \pi, \mathrm{Sym}^3(\rho_2), \psi) = \varepsilon(s, \pi \times \Pi, \psi) / \varepsilon(\varepsilon, \pi \otimes \omega, \psi), \tag{4.4}$$

where Π is the symmetric square lift of π in the sense of Gelbart–Jacquet [Ge-Ja] and ω denotes the central character of π. The root number $\varepsilon(s, \pi \times \Pi, \psi)$ is that of the Rankin–Selberg product attached to the pair (π, Π) (cf. [JPSS, Sh6, Sh7]).

In a recent monumental work [HT] Harris and Taylor proved the local Langlands conjecture for GL_n to the effect that it preserves root numbers for pairs. A simple proof of this was later given in [H2]. More precisely, if ϕ and Φ are the two and the three dimensional representations [H1, H2, HT, Ku] of the Deligne–Weil group parametrizing π and Π, respectively, then

$$\varepsilon(s, \phi \otimes \Phi, \psi) = \varepsilon(s, \pi \times \Pi, \psi).$$

Recall that $\Phi = \mathrm{Sym}^2 \phi$ which simply means $\mathrm{Sym}^2(\rho_2) \cdot \phi$, where $\mathrm{Sym}^2(\rho_2)$ is the three dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{C})$ on symmetric tensors of rank 2 (or homogeneous polynomials of degree 2 in 2 variables). Now using

$$\phi \otimes \mathrm{Sym}^2 \phi = \mathrm{Sym}^3 \phi \oplus (\phi \otimes \Lambda^2 \phi),$$

we have

$$\varepsilon(s, \phi \otimes \Phi, \psi) = \varepsilon(s, \mathrm{Sym}^3 \phi, \psi)\varepsilon(s, \phi \otimes \Lambda^2\phi, \psi)$$

from which

$$\varepsilon(s, \mathrm{Sym}^3 \phi, \psi) = \varepsilon(s, \pi, \mathrm{Sym}^3(\rho_2), \psi)$$

follows. We similarly recall $L(s, \pi, \mathrm{Sym}^3(\rho_2))$ from [Sh5]. It is again immediately clear that

$$L(s, \mathrm{Sym}^3 \phi) = L(s, \pi, \mathrm{Sym}^3(\rho_2))$$

(cf. [Sh5, BHK1, Sh6]). We collect this and the results from [Ki-Sh, Sh5] as:

THEOREM 4.2: (a) *Let π be an irreducible admissible representation of* $\mathrm{GL}_2(F)$, *where F is a local field of characteristic zero, archimedean or otherwise. Let ϕ: $W'_F \to \mathrm{GL}_2(\mathbb{C})$ be attached to π by the local Langlands correspondence* [Ku, La3, HT, H2]. *Then*

$$\varepsilon(s, \mathrm{Sym}^3 \phi, \psi) = \varepsilon(s, \pi, \mathrm{Sym}^3(\rho_2), \psi)$$

and

$$L(s, \mathrm{Sym}^3 \phi) = L(s, \pi, \mathrm{Sym}^3(\rho_2)).$$

(b) *Let π be a cuspidal representation of* $\mathrm{GL}_2(\mathbb{A})$, *where as before $\mathbb{A}$ is the ring of adeles of a number field. Fix a non-trivial character $\psi = \bigotimes_v \psi_v$ of $F\backslash\mathbb{A}$. Write $\pi = \bigotimes_v \pi_v$ and for each v, let ϕ_v: $W'_{F_v} \to \mathrm{GL}_2(\mathbb{C})$ parametrize π_v. Let $\varepsilon(s, \mathrm{Sym}^3 \phi_v, \psi_v)$ and $L(s, \mathrm{Sym}^3 \phi_v)$ be the corresponding Artin factors. Set*

$$L(s, \pi, \mathrm{Sym}^3(\rho_2)) = \prod_v L(s, \mathrm{Sym}^3 \phi_v)$$

and

$$\varepsilon(s,\pi,\mathrm{Sym}^3(\rho_2)) = \prod_v \varepsilon(s,\mathrm{Sym}^3\phi_v,\psi_v).$$

Then unless π is monomial, $L(s,\pi,\mathrm{Sym}^3(\rho_2))$ is entire. It satisfies

$$L(s,\pi,\mathrm{Sym}^3(\rho_2)) = \varepsilon(s,\pi,\mathrm{Sym}^3(\rho_2))L(1-s,\tilde{\pi},\mathrm{Sym}^3(\rho_2)).$$

Similarly we can consider $\varepsilon(s,\pi,\mathrm{Sym}^4(\rho_2),\psi)$ and $L(s,\pi,\mathrm{Sym}^4(\rho_2))$, where F is local and π is an irreducible admissible representation of $\mathrm{GL}_2(F)$. More precisely, we let

$$\varepsilon(s,\pi,\mathrm{Sym}^4(\rho_2),\psi) = \varepsilon(s,\Pi\times\Pi,\psi)/\varepsilon(s,\omega^2,\psi)\varepsilon(s,\Pi\otimes\omega,\psi), \tag{4.5}$$

where Π is the symmetric square lift of π defined before and ω is the central character of π. If π is parametrized by ϕ as before we have

$$\varepsilon(s,\mathrm{Sym}^4\phi,\psi) = \varepsilon(s,\mathrm{Sym}^2\phi\otimes\mathrm{Sym}^2\phi,\psi)/\varepsilon(s,(\Lambda^2\phi)^2,\psi)\varepsilon(s,\mathrm{Sym}^2\phi\otimes\Lambda^2\phi,\ \psi).$$

Again appealing to [HT, H2] we have

$$\varepsilon(s,\pi,\mathrm{Sym}^4(\rho_2),\psi) = \varepsilon(s,\mathrm{Sym}^4\phi,\psi).$$

Similarly for L-functions. We have [Sh6, Sh7] (see Remark 2).

PROPOSITION 4.3: *Part* (a) *of Theorem 4.2 is valid if* $\mathrm{Sym}^3(\rho_2)$ *and* $\mathrm{Sym}^3\phi$ *are replaced with* $\mathrm{Sym}^4(\rho_2)$ *and* $\mathrm{Sym}^4\phi$. *If $\pi = \bigotimes_v \pi_v$ is a cusp form on* $\mathrm{GL}_2(\mathbb{A})$ *and*

$$L(s,\pi,\mathrm{Sym}^4(\rho_2)) = \prod_v L(s,\mathrm{Sym}^4\phi_v),$$

and

$$\varepsilon(s,\pi,\mathrm{Sym}^4(\rho_2)) = \prod_v \varepsilon(s,\mathrm{Sym}^4\phi_v,\psi_v),$$

then $L(s,\pi,\mathrm{Sym}^4(\rho_2))$ is holomorphic on $\mathrm{Re}(s)\geq 1$ *unless either π, or Π, is monomial (see Remark 2), and extends to a meromorphic function of s on $\mathbb{C}$. It satisfies*

$$L(s,\pi,\mathrm{Sym}^4(\rho_2)) = \varepsilon(s,\pi,\mathrm{Sym}^4(\rho_2))L(1-s,\tilde{\pi},\mathrm{Sym}^4(\rho_2)).$$

Remark 1: In view of (4.4) and (4.5) and the main result of [BHK2], one can now also compute conductors for Sym^3 and Sym^4 root numbers.

Remark 2: Lemma 9.2 of [Sh 7] is incorrect and therefore it is possible for Π to satisfy $\Pi \cong \Pi \otimes \eta$ for a non-trivial (cubic) character η, which is to say Π is dihedral or monomial. In that case, i.e., when $\Pi \cong \Pi \otimes \eta$, $\eta \neq 1$, as it is argued in the proof of Lemma 9.2, π itself will have a cubic cyclic base change, with the extension defined by η, which is monomial.

Remark 3: The first author wishes to make the following corrections to [Ki3]: (1) In Proposition 0.1 and Proposition 2.1, "$w_0\sigma = \sigma$" should be replaced with "$w_0\sigma \simeq \sigma$". (2) On page 840, line -4, "$v \notin S$" should be replaced with "$v \in S$". (3) On page 841, line -5, "w_0" should be "w_l".

References

[B] D. Blasius, *Critical values of certain tensor product L-functions*, Inventiones Mathematicae **90** (1987), 181–188.

[BHK1] C. Bushnell, G. Henniart and P. Kutzko, *Correspondance de Langlands locale pour GL_n et conducteurs de pairs*, preprint, 1997.

[BHK2] C. Bushnell, G. Henniart and P. Kutzko, *Local Rankin–Selberg convolutions for* GL_n*: Explicit conductor formula*, Journal of the American Mathematical Society (to appear).

[C-Sh] W. Casselman and F. Shahidi, *On irreducibility of standard modules for generic representations*, Annales Scientifiques de l'École Normale Supérieure **31** (1998), 561–589.

[D] P. Deligne, *Valeurs de fonctions L et périodes d'intgrales*, Proceedings of Symposia in Pure Mathematics II **33** (1979), 313–346.

[Ga] P. Garrett, *Decomposition of Eisenstein series, Rankin triple products*, Annals of Mathematics **125** (1987), 209–235.

[GH] P. Garrett and M. Harris, *Special values of triple product L-functions*, American Journal of Mathematics **115** (1993), 159–238.

[Ge-Ja] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Annales Scientifiques de l'École Normale Supérieure **11** (1978), 471–552.

[HK] M. Harris and S. Kudla, *The central critical value of a triple product L-function*, Annals of Mathematics **133** (1991), 605–672.

[HT] M. Harris and R. Taylor, *On the geometry and cohomology of some simple Shimura varieties*, Preliminary version (1998).

[H1] G. Henniart, *La conjecture de Langlands locale pour* GL(3), Mémoires de la Société Mathématique de France **11/12** (1984), 1–186.

[H2] G. Henniart, *Une preuve simple de conjectures de Langlands pour* GL(n) *sur un corp* p*-adiques*, Inventiones Mathematicae (to appear).

[Ik] T. Ikeda, *On the location of poles of the triple L-functions*, Compositio Mathematica **83** (1992), 187–237.

[JPSS] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Rankin–Selberg convolutions*, American Journal of Mathematics **105** (1983), 367–464.

[Ku] P. Kutzko, *The Langlands conjecture for* GL_2 *of a local field*, Annals of Mathematics **112** (1980), 381–412.

[Ki1] H. Kim, *The residual spectrum of* Sp_4, Compositio Mathematica **99** (1995), 129–151.

[Ki2] H. Kim, *The residual spectrum of* G_2, Canadian Journal of Mathematics **48** (1996), 1245–1272.

[Ki3] H. Kim, *Langlands–Shahidi method and poles of automorphic L-functions: application to exterior square L-functions*, Canadian Journal of Mathematics **51** (1999), 835–849.

[Ki4] H. Kim, *Langlands–Shahidi method and poles of automorphic L-functions II*, Israel Journal of Mathematics **117** (2000), 261–284; *Correction*, **118** (2000), 379.

[Ki-Sh] H. Kim and F. Shahidi, *Symmetric cube L-functions of* GL_2 *are entire*, Annals of Mathematics **150** (1999), 645–662.

[L-La] J-P. Labesse and R. P. Langlands, *L-indistinguishability for* SL(2), Canadian Journal of Mathematics **31** (1979), 726–785.

[La1] R. P. Langlands, *Euler Products*, Yale University Press, 1971.

[La2] R. P. Langlands, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Mathematics **544**, Springer-Verlag, Berlin, 1976.

[La3] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups* (P. J. Sally, Jr. and D. A. Vogan, eds.), Mathematical Surveys and Monographs **31** (1989), 101–170.

[Li] J. S. Li, *Some results on the unramified principal series of* p*-adic groups*, Mathematische Annalen **292** (1992), 747–761.

[M-W1] C. Moeglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series, une paraphrase de l'Ecriture*, Cambridge Tracts in Mathematics, Vol. 113, Cambridge University Press, 1995.

[M-W2] C. Moeglin and J.-L. Waldspurger, *Le spectre résiduel de* GL(n), Annales Scientifiques de l'École Normale Supérieure **22** (1989), 605–674.

[Mu] G. Muić, *The unitary dual of* p*-adic* G_2, Duke Mathematical Journal **90** (1997), 465–493.

[O] T. Orloff, *Special values and mixed weight triple products*, Inventiones Mathematicae **90** (1987), 169–180.

[PS-Ra] I. Piatetski-Shapiro and S. Rallis, *Rankin triple L-functions*, Compositio Mathematica **64** (1987), 31–115.

[Ram1] D. Ramakrishnan, *On the coefficients of cusp forms*, Mathematical Research Letters **4** (1997), 295–307.

[Ram2] D. Ramakrishnan, *Modularity of the Rankin–Selberg L-series, and multiplicity one for* SL(2), preprint (1998).

[Sh1] F. Shahidi, *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Annals of Mathematics **132** (1990), 273–330.

[Sh2] F. Shahidi, *Fourier transforms of intertwining operators and Plancherel measures for* $\mathrm{GL}(n)$, American Journal of Mathematics **106** (1984), 67–111.

[Sh3] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Annals of Mathematics **127** (1988), 547–584.

[Sh4] F. Shahidi, *Automorphic L-functions: a survey*, in *Automorphic Forms, Shimura Varieties, and L-functions* (L. Clozel and J. Milne, eds.), Vol. II (Ann Arbor, MI, 1988), Academic Press, New York, 1990, pp. 415–437.

[Sh5] F. Shahidi, *Third symmetric power L-functions for* GL(2), Compositio Mathematica **70** (1989), 245–273.

[Sh6] F. Shahidi, *Local coefficients as Artin factors for real groups*, Duke Mathematical Journal **52** (1985), 973–1007.

[Sh7] F. Shahidi, *Symmetric power L-functions*, in *Elliptic Curves and Related Topics* (H. Kisilevsky and R. Murty, eds.), CRM Proceedings and Lecture Notes, American Mathematical Society, Providence, NJ, 1994, pp. 159–182.

[S] G. Shimura, *On the periods of modular forms*, Mathematische Annalen **229** (1977), 211–221.

[T] J. Tate, *Number theory background*, Proceedings of Symposia in Pure Mathematics II **33** (1970), 3–26.

[Z] D. Zagier, *Modular forms whose Fourier coefficients involve zeta functions of quadratic fields*, in *Modular Forms of One Variable VI*, Lecture Notes in Mathematics **627**, Springer-Verlag, New York, 1977, pp. 105–169.

[Za] S. Zampera, *The residual spectrum of type* G_2, Journal de Mathématiques Pures et Appliquées **76** (1997), 805–835.

[Zh] Y. Zhang, *The holomorphy and nonvanishing of normalized intertwining operators*, Pacific Journal of Mathematics **180** (1997), 385–398.